

- (1) (15 marks) Find the second order Taylor approximation of f at $(0, 0, 0)$, where

(a) $f(x, y, z) = \frac{1}{1-xyz}$.

(b) $f(x, y, z) = e + x^2 + y^2 + z^2 + xy + xy^2e^z + x^3yz + x^2y^2z^2$.

- (2) (15 marks) Identify all critical points of the function

$$f(x, y) = x^3 - y^3 - 3xy,$$

and then classify each as the location of a local maximum, local minimum, or saddle point.

- (3) (15 marks) Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = |xy|.$$

Which of the following statements are true (justify your answer)?

- (a) f is continuous at $(0, 0)$.
 - (b) Both $f_x(0, 0)$ and $f_y(0, 0)$ exist.
 - (c) f is differentiable at $(0, 0)$.
 - (d) $f_x(0, r)$ does not exist for any $r \neq 0$.
- (4) (15 marks) Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } x \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Which of the following statements are true (justify your answer)?

- (a) f_x and f_y are continuous on $\mathbb{R}^2$.
 - (b) f_{xy} and f_{yx} are continuous on $\mathbb{R}^2$.
 - (c) $f_{xy}(0, 0) = f_{yx}(0, 0)$.
- (5) (15 marks) Let $\mathcal{O}_n$ be an open subset of $\mathbb{R}^n$ and let $a \in \mathcal{O}_n$. Prove that there is no function $f : \mathcal{O}_n \rightarrow \mathbb{R}$ such that the directional derivative

$$(D_v f)(a) > 0,$$

for every $v \in \mathbb{R}^n \setminus \{0\}$.

- (6) (15 marks) Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^1 function and suppose $F(0) = 0$. Prove that there exist continuous functions $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \dots, n$, such that

$$F(x) = \sum_{i=1}^n x_i f_i(x).$$

- (7) (20 marks) State the implicit function theorem. Then, use the implicit function theorem to prove that the exponential function cannot have zeros.